

## EVOLUTIONARY MODEL OF FINITE-STRAIN THERMOELASTICITY

A. A. Rogovoi and O. S. Stolbova

UDC 539.3

*The equation of state of finite-strain thermoelasticity is obtained using a formalized approach to constructing constitutive relations for complex media under the assumption of closeness of intermediate and current configurations. A variational formulation of the coupled thermoelastic problem is proposed. The constitutive equation, the heat-conduction equation, the relations for internal energy, free energy, and entropy, and the variational formulation of the coupled problem of finite-strain thermoelasticity are tested on the problem of uniaxial extension of a bar. The model adequately describes experimental data for elastomers, such as entropic elasticity, temperature inversion, and temperature variation during an adiabatic process.*

**Key words:** *thermoelasticity, finite strains, slight compressibility, constitutive equations, heat-conduction equation, testing of model.*

**Introduction.** In [1–5], a decomposition of the total site gradient into elastic, inelastic, and temperature gradients, which is similar in form to the well-known Lie expansion but free from the drawbacks of the latter, was constructed within the framework of the kinematics determined by superposition of elastic–inelastic site gradients (which transform an intermediate configuration to a close current configuration) on final elastic–inelastic gradients (which transform the initial configuration to an intermediate). Based on the laws of thermodynamics and the principle of objectivity, it has been shown that the inelastic and temperature site gradients should be pure deformations without rotations. In view of this requirement, the missing relationship was obtained between small inelastic strains  $e_{IN}$  with known constitutive relations, and small inelastic rotations  $d_{IN}$ , and also between small temperature strains  $e_{\Theta}$  with known constitutive relations and small temperature rotations  $d_{\Theta}$ . Stress and entropy relations following from thermodynamics were obtained and the heat-conduction equation is constructed. The equations of state for finite thermo-elastic–inelastic deformations were written for an intermediate configuration close to the current one. In the fourth-rank tensor determining the material properties in an intermediate configuration and dependent only on elastic kinematics, the constants were assumed to be functions of temperature and scalar structural parameters determined by inelastic kinematics.

In the present work, a model for thermoelastic behavior at finite strains is constructed using the theoretical notions listed above and is tested on a simple problem, for which there is a large body of experimental data.

As noted in [6], finite-strain thermoelasticity is a classical and elaborate theory of nonlinear continuum mechanics. Fundamental papers give a thermodynamic foundation of the kinematics proposed to describe the thermoelastic process (see, for example, [6–8] and the bibliography therein). In the present work, thermoelastic relations are obtained using the general approach to constructing the constitutive equations of elastic–inelastic media at finite strains [1–5], which is based on the kinematics constructed under the assumption of closeness of intermediate and current configurations and allow a description of elastoplastic process [1] and viscoelastic process [2]. Elastic behavior is described by a relation in which both slight compressibility and incompressibility and the dependence of the three-dimensional modulus and shear modulus on the volume variation are taken into account via one of the generalized elastic moduli [9–12].

---

Institute of Mechanics of Continuous Media, Ural Division, Russian Academy of Sciences, Perm' 614013; rogovoy@icmm.ru; sos@kig.pstu.ac.ru. Translated from *Prikladnaya Mekhanika i Tekhnicheskaya Fizika*, Vol. 49, No. 3, pp. 184–196, May–June, 2008. Original article submitted February 15, 2007; revision submitted June 7, 2007.

**1. Constitutive Thermoelastic Equation and Heat-Conduction Equation.** In [1–3], equivalent forms of the equations of state for finite elastic–inelastic strains with respect to an intermediate configuration close to the current configuration are constructed using the kinematics determined by the superposition of small strains (site gradients) on finite strains for the case where the constants of the elastic potential do not depend on inelastic strains. We use the approach developed in [1–3] to construct the equation of state for a thermoelastic material in an intermediate configuration. In [3], the true-stress tensor is represented as

$$T = 4J^{-1}F \cdot \left[ \int_0^t \left( F \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \cdot F^t \right) \cdot D_E d\tau \right] \cdot F^t. \quad (1.1)$$

Here  $F$  is the total (in this case, thermoelastic) site gradient,  $J = I_3(F)$  is the third invariant  $F$  (the Jacobian determining the relative volume variation),  $C_E$  is the Cauchy–Green elastic strain measure,  $D_E = \dot{e}_E$  is the elastic displacement rate deformation, which, in this case, coincides with the elastic strain rate,  $W$  is the elastic potential, whose constants for thermoelastic processes, according to [3], depend on the absolute temperature  $\Theta$  as on a parameter:  $\Theta = \Theta(t)$ . The notation  $A \overset{3}{\circ} B^{IV}$  denotes the scalar premultiplication of the second-rank tensor  $A$  by the third basis vector of the fourth-rank tensor  $B^{IV}$ . According to [3–5], the kinematic tensors are given by the expressions

$$F = F_E \cdot F_\Theta = (g + \varepsilon h) \cdot F_* = [g + \varepsilon(h_E + h_\Theta)] \cdot F_*, \quad F_* = F_{E*} \cdot F_{\Theta*}. \quad (1.2)$$

Here and below, the quantities with the subscript “\*” correspond to the intermediate configuration (time  $t_*$ ), and the quantities without it to the close current configuration (time  $t$ ); the closeness of these configurations is characterized by the small positive quantity  $\varepsilon$ ,  $g$  is the unit tensor,  $h = h_E + h_\Theta$ ,  $h_E$ , and  $h_\Theta$  are the gradients of the vectors of the small total, elastic, and temperature displacements with respect to the intermediate configuration [these gradients are represented as the sum of symmetric gradients  $e$ ,  $e_E$ , and  $e_\Theta$  (small total, elastic, and temperature strains) and skew-symmetric gradients  $d$ ,  $d_E$ , and  $d_\Theta$  (small total, elastic, and temperature rotations), such that  $e = e_E + e_\Theta$  and  $d = d_E + d_\Theta$ ];  $C_E = F_E^t \cdot F_E$  and  $F_E$  and  $F_\Theta$  are the elastic and temperature site gradients, respectively, written as

$$F_E = (g + \varepsilon h_E) \cdot F_{E*}, \quad F_\Theta = (g + \varepsilon F_{E*}^{-1} \cdot h_\Theta \cdot F_{E*}) \cdot F_{\Theta*}.$$

The gradient  $F_\Theta$  corresponds to pure deformation without rotation, i.e., in the polar decomposition  $F_\Theta = R_\Theta \cdot U_\Theta$ , the orthogonal tensor  $R_\Theta = g$ . As shown in [5], assuming that  $e_\Theta$  is given by the linear temperature-expansion law  $e_\Theta = \beta\theta g$ , where  $\beta$  is the linear temperature-expansion coefficient and  $\theta$  is a small temperature variation, we have  $d_\Theta = 0$ . Since  $d = d_E + d_\Theta$ , we have  $d_E = d$ . As a result, we obtain

$$F_E = [g + \varepsilon(h - \beta\theta)] \cdot F_{E*}, \quad F_\Theta = U_\Theta = (1 + \varepsilon\beta\theta)U_{\Theta*}. \quad (1.3)$$

In relation (1.1), the integral from 0 to  $t$  is represented as two integrals: from 0 to  $t_*$  and from  $t_*$  to the time  $t$  close to  $t_*$ , and the temperature, which is one of the arguments of the elastic potential  $W$  and depends on the current time  $t$ , is represented as  $\Theta = \Theta_* + \varepsilon\theta$ . Then,

$$\frac{\partial^2 W(\Theta)}{\partial C_E^2} = \frac{\partial^2 W(\Theta_* + \varepsilon\theta)}{\partial C_E^2} = \frac{\partial^2 W(\Theta_*)}{\partial C_E^2} + \varepsilon\theta \frac{\partial^3 W(\Theta)}{\partial \Theta \partial C_E^2} \Big|_{\Theta=\Theta_*}.$$

As a result, using relations (1.2) and proceeding in the same way as in the derivation of expression (4.7) in [3], we write the constitutive equation (1.1) for the intermediate configuration as

$$T = [1 - \varepsilon I_1(e)]T_* + \varepsilon h \cdot T_* + \varepsilon T_* \cdot h^t + \varepsilon\theta(T, \Theta)_* + \varepsilon \tilde{L}_6^{IV} \cdot e_E. \quad (1.4)$$

This approximate equation reduces to the evolutionary differential equation with the Truesdell objective derivative

$$T^{\text{Tr}} = \dot{\theta} T_{,\Theta} + L_6^{IV} \cdot \dot{e}_E, \quad (1.5)$$

where  $\dot{\theta} = \dot{\Theta}$ . In relation (1.4),  $T_*$  and  $(T, \Theta)_*$  are the true stresses and their derivatives with respect to temperature in the intermediate configuration, respectively,  $I_1$  is the first invariant, and  $\tilde{L}_6^{IV}$  is a fourth-rank tensor that defines the response of the material to small elastic strains with respect to the intermediate configuration:

$$\tilde{L}_6^{IV} = 4J_*^{-1}F_* \cdot \left( F_* \overset{3}{\circ} \frac{\partial^2 W}{\partial C_E^2} \Big|_{\substack{C_E=C_{E*} \\ \Theta=\Theta_*}} \overset{2}{*} F_*^t \right) \cdot F_*^t, \quad (1.6)$$

the notation  $B^{IV} \overset{2}{*} A$  denotes the scalar multiplication on the right of the second-rank tensor  $A$  by the second basis vector of the fourth-rank tensor  $B^{IV}$ . In relation (1.5), the fourth-rank tensor  $L_6^{IV}$  defines the response of the material to the elastic strain rates with respect to the current configuration and is written in the form of (1.6), where all kinematic quantities and temperature determined in the current rather than intermediate configuration.

It is easy to show that, in expression (1.5), the tensor  $T_{,\Theta} \equiv \partial T / \partial \Theta$  is represented as

$$T_{,\Theta} = [1 - \varepsilon I_1(e)](T_{,\Theta})_* + \varepsilon h \cdot (T_{,\Theta})_* + \varepsilon (T_{,\Theta})_* \cdot h^t + \varepsilon \theta (T_{,\Theta\Theta})_* + \varepsilon \tilde{L}_{6,\Theta}^{IV} \cdot e_E, \quad (1.7)$$

$$\tilde{L}_{6,\Theta}^{IV} = 4J_*^{-1} F_* \cdot \left( F_* \overset{3}{\circ} \frac{\partial^3 W}{\partial \Theta \partial C_E^2} \Big|_{\substack{C_E = C_{E*} \\ \Theta = \Theta_*}} \overset{2}{*} F_*^t \right) \cdot F_*^t.$$

Expression (1.7) is similar to (1.4). The structure of relations (1.4), (1.6), and (1.17) shows how it is possible to construct higher-order derivatives of the stress with respect to temperature.

To describe finite strains of a medium which, in the initial state, is an isotropic slightly compressible elastic material, we use the elastic potential proposed in [9–12]:

$$W = \hat{W} + \sigma(J_E^2 - 1) - \alpha(\sigma - \chi_1)^2/2, \quad J_E = I_{3E}(F_E).$$

Here

$$\begin{aligned} \hat{W} &= k_1(\hat{I}_1 - 3) + k_2(\hat{I}_2 - 3), & \sigma &= \chi_1 + \chi_2(J_E^2 - 1), \\ \chi_1 &= p_1(\hat{I}_1 - 3) + p_2(\hat{I}_2 - 3), & \chi_2 &= \chi_{20} + q_1(\hat{I}_1 - 3) + q_2(\hat{I}_2 - 3), \\ \hat{I}_1 &= I_{1E} - (I_{3E} - 1), & \hat{I}_2 &= I_{2E} - 2(I_{3E} - 1), & \alpha &= 1/\chi_2, \end{aligned}$$

$I_{1E}$ ,  $I_{2E}$ , and  $I_{3E} = J_E^2$  are invariants of the tensor  $C_E = F_E^t \cdot F_E$ ,  $k_1$ ,  $k_2$ ,  $p_1$ ,  $p_2$ ,  $q_1$ ,  $q_2$ , and  $\chi_{20}$  are material parameters, which, as shown in [9–12] obey the relations  $2(k_1 + k_2) = G$ ,  $p_i = p_{i0}G$ , and  $q_i = q_{i0}G$  for  $i = 1, 2$  ( $G$  is the shear modulus). Then, in view of relation (1.6), the last term in (1.4) can be written as

$$\begin{aligned} \tilde{L}_6^{IV} \cdot e_E &= 4J_*^{-1} \{ (c_1 - \sigma_*) I_{3E*} [Y \cdot e_E \cdot Y - Y(Y \cdot e_E)] \\ &+ I_{3E*} [Y(\Phi_* \cdot e_E) + (\Phi_* - 2I_{3E*} Y)(Y \cdot e_E)] G [p_{10} + q_{10}(I_{3E*} - 1)] \\ &+ [\Phi_* (\Phi_* \cdot e_E) - \Phi_* \cdot e_E \cdot \Phi_* + 2I_{3E*} (Y \cdot e_E \cdot Y - (Y \cdot e_E) Y)] c_2 + I_{3E*} [I_{1E*} Y(\Phi_* \cdot e_E) - Y(X \cdot e_E)] \\ &+ (I_{1E*} \Phi_* - X - 4I_{3E*} Y)(Y \cdot e_E) \} G [p_{20} + q_{20}(I_{3E*} - 1)] + I_{3E*}^2 [\chi_{20} + G(q_{10}(\hat{I}_{1E*} - 3) + q_{20}(\hat{I}_{2E*} - 3))] Y(Y \cdot e_E), \\ c_i &= k_i + G[p_{i0} + (1/2)q_{i0}(I_{3E*} - 1)](I_{3E*} - 1) \quad (i = 1, 2), \end{aligned} \quad (1.8)$$

$$\sigma_* = \chi_{20}(I_{3E*} - 1) + G\{ [p_{10} + q_{10}(I_{3E*} - 1)](\hat{I}_{1E*} - 3) + [p_{20} + q_{20}(I_{3E*} - 1)](\hat{I}_{2E*} - 3) \},$$

where  $Y = F_* \cdot C_{E*}^{-1} \cdot F_*^t$ ,  $X = F_* \cdot C_{E*} \cdot F_*^t$ , and  $\Phi_* = F_* \cdot F_*^t$  is the Finger strain measure tensor, whose invariants coincide with the corresponding invariants of the tensor  $C_*$ . In relation (1.8), the small elastic strains are represented as the difference of the small total and temperature strains  $e_E = e - e_\Theta$ , and this completes the construction of the constitutive equation (1.4) for the thermoelastic behavior of materials at finite elastic (slightly compressible material) and temperature strains.

We assume that the shear modulus  $G$  (and, hence, the quantities  $p_1$ ,  $p_2$ ,  $q_1$ , and  $q_2$ ), through the material parameters  $k_1$  and  $k_2$ , and the quantity  $\chi_{20}$  depend on temperature. We represent this relation in the general form

$$\begin{aligned} \gamma_i &= \gamma_{i0} + \int_{\Theta_0}^{\Theta} \gamma_{i1}(\Theta_1) d\Theta_1 = \gamma_{i*} + \varepsilon \gamma_{i1*} \theta, & \theta &= \Theta - \Theta_*, \\ \gamma_{i*} &= \gamma_{i0} + \int_{\Theta_0}^{\Theta_*} \gamma_{i1}(\Theta_1) d\Theta_1, & \gamma_{i1*} &= \gamma_{i1}(\Theta_*) \end{aligned} \quad (1.9)$$

( $\Theta_0$  is the absolute temperature at the initial time for the entire process). From this, it follows that

$$\begin{aligned}\gamma_{i,\Theta} &= \gamma_{i1}(\Theta) = \gamma_{i1}(\Theta_* + \varepsilon\theta) = \gamma_{i1*} + \varepsilon\gamma_{i2*}\theta, & \gamma_{i2*} &= \gamma_{i1,\Theta}\Big|_{\Theta=\Theta_*}, \\ \gamma_{i,\Theta\Theta} &= \gamma_{i1,\Theta} = \gamma_{i2}(\Theta_* + \varepsilon\theta) = \gamma_{i2*} + \varepsilon\gamma_{i3*}\theta, & \gamma_{i3*} &= \gamma_{i2,\Theta}\Big|_{\Theta=\Theta_*},\end{aligned}\quad (1.10)$$

etc. Let  $\gamma_1 = k_1$ ,  $\gamma_2 = k_2$ ,  $\gamma_3 = \chi_{20}$ , and  $\gamma_4 = G = 2(k_1 + k_2) = 2(\gamma_1 + \gamma_2)$ . Then, in relation (1.6), the constants  $k_1$ ,  $k_2$ ,  $\chi_{20}$ , and  $G$  included in expressions (1.8) need to be replaced, according to (1.9), by the quantities  $k_{1*}$ ,  $k_{2*}$ ,  $\chi_{20*}$ , and  $G_*$ ; in relation (1.7), they need to be replaced, according to (1.10), by the quantities  $k_{11*}$ ,  $k_{21*}$ ,  $\chi_{201*}$ ,  $G_{1*}$ , etc. This makes it possible to calculate the tensors  $T$  (1.4) and  $T_{,\Theta}$  (1.7) and higher-order derivatives of the stress with respect to temperature using a unified algorithm.

Similarly to the recursive relations (1.4) and (1.7), we construct a recursive relation for the functional  $W_1$  obtained in [3, 4]:

$$\begin{aligned}W_1 &= \int_0^t JT(\Theta) \cdot D_E d\tau = \int_0^{t_*} JT(\Theta_* + \varepsilon\theta) \cdot D_E d\tau + \int_{t_*}^t JT(\Theta) \cdot D_E d\tau \\ &= \int_0^{t_*} J[T(\Theta_*) + \varepsilon T_{,\Theta}(\Theta_*)\theta] \cdot D_E d\tau + \varepsilon J_* T_* \cdot e_E.\end{aligned}$$

From this, we have the relation

$$W_1 = W_{1*} + \varepsilon\theta(W_{1,\Theta})_* + \varepsilon J_* T_* \cdot e - \varepsilon\beta_*\theta J_* I_1(T_*), \quad (1.11)$$

and the following relation, which is easily obtained and extended to higher-order derivatives with respect to temperature

$$W_{1,\Theta} = (W_{1,\Theta})_* + \varepsilon\theta(W_{1,\Theta\Theta})_* + \varepsilon J_*(T_{,\Theta})_* \cdot e - \varepsilon\beta_*\theta J_* I_1((T_{,\Theta})_*). \quad (1.12)$$

Relations (1.11) and (1.12) were derived taking into account that  $e_E = e - e_\Theta = e - \beta\theta g$ .

In the case of thermoelasticity, the heat-conduction equation obtained in [4] [relation (2.12)] has the form

$$\begin{aligned}A\dot{\Theta} + B\Theta &= \rho\dot{\Omega} + \tilde{\nabla} \cdot (\lambda\tilde{\nabla}\Theta), \\ A &= \Theta\left(\beta_{,\Theta}I_1(T) + \beta I_1(T_{,\Theta}) - J^{-1}W_{1,\Theta\Theta}\right) + J^{-1}\rho_0 c_T, \\ B &= \beta(I_1(T)I_1(D) + I_1(\dot{T})) - T_{,\Theta} \cdot D.\end{aligned}\quad (1.13)$$

Here  $\rho_0$  and  $\rho$  are the density of the material in the undeformed and current configurations,  $c_T$  is the heat capacity at zero stress,  $\dot{\Omega}$  is the rate of heat production by internal sources in unit mass,  $\lambda$  is the thermal conductivity, and  $\tilde{\nabla}$  is the Hamiltonian for the current configuration. In [4], the expressions for  $A$  and  $B$  were derived from relation (2.12) taking into account that

$$\frac{d}{dt}(W_{1,\Theta}) = \dot{W}_{1,\Theta\Theta} + JT_{,\Theta} \cdot D - \beta\dot{\theta} J I_1(T_{,\Theta}). \quad (1.14)$$

The last relation is obtained from Eq. (1.12) written as

$$\Delta W_{1,\Theta} = W_{1,\Theta} - (W_{1,\Theta})_* = (W_{1,\Theta\Theta})_* \Delta\Theta + J_*(T_{,\Theta})_* \cdot \Delta e - \beta_* J_* I_1((T_{,\Theta})_*) \Delta\Theta$$

( $\Delta W_{1,\Theta}$  is the increment of  $W_{1,\Theta}$ ,  $\Delta\Theta = \varepsilon\theta$  is the temperature increment  $\Theta$ , and  $\Delta e = \varepsilon e$  is the strain increment) by dividing its left and right sides by the time increment  $\Delta t$  and passing to the limit as  $\Delta t \rightarrow 0$  under the assumption that all limits exist.

The boundary conditions for the heat-conduction equation (1.13) are written as

$$\Theta\Big|_{S_\Theta} = \Theta_S, \quad \mathbf{N} \cdot \mathbf{q}\Big|_{S_q} = q_n, \quad S = S_\Theta \cup S_q,$$

where  $\mathbf{N}$  is the outward unit normal to the surface in the current configuration,  $S$  is the total surface of the body in the current configuration, and the parts of the surface  $S_\Theta$  and  $S_q$  can be empty sets. Assuming that the heat flux  $\mathbf{q}$  obeys the Fourier law  $\mathbf{q} = -\lambda\tilde{\nabla}\Theta$ , we find the boundary conditions of the first and second kinds

$$\Theta \Big|_{S_\Theta} = \Theta_S, \quad -\mathbf{N} \cdot \lambda \tilde{\nabla} \Theta \Big|_{S_q} = q_n. \quad (1.15)$$

Setting  $q_n = \alpha_s(\Theta - \Theta_c)$  ( $\alpha_s$  is the heat-transfer coefficient and  $\Theta_c$  is the absolute ambient temperature), we obtain boundary conditions of the third kind. The initial conditions for the heat-conduction equation are  $\Theta(\mathbf{x}, t_0) = \Theta_0(\mathbf{x})$ .

**2. Variational Formulation of the Boundary-Value Thermoelastic Problem.** The relations describing quasistatic thermoelastic processes contain an equilibrium equation, kinematic and constitutive equations, boundary conditions (generally, mixed) for displacements on the surface  $S_u$  and stresses on the surface  $S_p$ , and the heat-conduction equation (1.13) with boundary (1.15) and initial conditions. Applying the standard Galerkin procedure to the equilibrium and heat-conduction equations and the boundary conditions for stresses and temperature in the form (1.15) and taking into account the constraints imposed on the displacements on the surface  $S_u$  and on the temperature on the surface  $S_\Theta$  provided that displacement and temperature variations are independent of each other, we obtain the well-known weak (variational) coupled formulation of the thermoelastic problem in the generalized Lagrange form for the initial configuration and any time  $t$ :

$$\begin{aligned} & \frac{1}{2} \int_{V_0} P_{II} \cdot \delta C \, dV_0 - \int_{V_0} \rho_0 \mathbf{f} \cdot \delta \mathbf{u} \, dV_0 - \int_{S_0} J_s \mathbf{p} \cdot \delta \mathbf{u} \, dS_0 = 0, \\ & \int_{V_0} J[\lambda \nabla \Theta \cdot C^{-1} \cdot \delta(\nabla \Theta) + (A\dot{\Theta} + B\Theta)\delta\Theta] \, dV_0 - \int_{V_0} \rho_0 \dot{\Omega} \delta \Theta \, dV_0 + \int_{S_0} J_s q_n \delta \Theta \, dS_0 = 0. \end{aligned} \quad (2.1)$$

Here  $V_0$  and  $S_0$  are the volume and total surface of the body in the initial configuration,  $P_{II}$  is the Piola–Kirchhoff symmetric stress tensor,  $C$  is the Cauchy–Green total strain measure,  $\mathbf{f}$ ,  $\mathbf{p}$  are the mass and surface forces,  $\mathbf{u}$  is the total displacement vector,  $J_s = J\sqrt{\mathbf{n} \cdot C^{-1} \cdot \mathbf{n}}$  is the Jacobian which defines the relative change of the current and initial surfaces,  $\mathbf{n}$  is the outward unit normal to the surface in the initial configuration, and  $\nabla$  is the Hamiltonian in the initial configuration. Setting  $P_{II} = JF^{-1} \cdot T \cdot F^{-t}$ , from relation (1.4) we obtain

$$P_{II} = (P_{II})_* + \varepsilon \theta (P_{II, \Theta})_* + \varepsilon J_* F_*^{-1} \cdot (\tilde{L}_6^{IV} \cdot e_E) \cdot F_*^{-t}. \quad (2.2)$$

**3. Testing of the Model.** The constitutive equation and the heat-conduction equations constructed in the present paper and the free-energy and entropy relations obtained in [4] were tested on a simple problem of bar tension (uniaxial state of stress), for which there are many experimental data. We consider only those processes in which the stress and the temperature fields are uniform. Although the fields are uniform, we solve the problem using the variational equations (2.1), which provides further testing of the equations.

The extension process of an initially isotropic rectilinear bar along the  $z$  axis is divided into a number of small steps. Then, in Cartesian coordinates, the positions of a point in the  $n$ th step (in the intermediate configuration) and in the  $(n+1)$ th step (in the current configuration close to the intermediate configuration) are given by the radius vectors

$$\mathbf{R}_* = \alpha_{1*}(xi + yj) + \alpha_{2*}zk, \quad \mathbf{R} = \alpha_1(xi + yj) + \alpha_2zk.$$

Here the basis vector  $\mathbf{k}$  is directed along the bar, and other two basis vectors lie in the plane of its cross section;  $\alpha_1 = \alpha_{1*} + \varepsilon\xi$  and  $\alpha_2 = \alpha_{2*} + \varepsilon\eta$ , where  $\alpha_{1*}$  and  $\alpha_{2*}$  are the relative elongations of the bar in the transverse and axial directions in the current and intermediate configurations, respectively, and  $\xi$  and  $\eta$  are their increments. In this case,  $\alpha_{2*}$  and  $\eta$  are specified quantities and  $\alpha_{1*}$  is known from the solution of the problem in the previous step. As above, the closeness of the intermediate and current configurations is determined by the small positive parameter  $\varepsilon$ . From these relations, we find the displacement vector which transforms the intermediate to the current configuration:

$$\mathbf{u} = \mathbf{R} - \mathbf{R}_* = \xi(xi + yj) + \eta zk.$$

Using the standard procedure, we construct the Hamiltonian for the intermediate configuration:

$$\overset{*}{\nabla} = \frac{1}{\alpha_{1*}} \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} \right) + \frac{1}{\alpha_{2*}} \mathbf{k} \frac{\partial}{\partial z}$$

and determine the total small strain tensor that arises in transition from the intermediate to the current configuration:

$$e = \frac{\xi}{\alpha_{1*}} (\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + \frac{\eta}{\alpha_{2*}} \mathbf{k}\mathbf{k}.$$

The site gradient is represented as

$$F = (\nabla \mathbf{R})^t = F_* + \varepsilon[\xi(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + \eta\mathbf{k}\mathbf{k}], \quad F_* = \alpha_{1*}(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + \alpha_{2*}\mathbf{k}\mathbf{k}, \quad (3.1)$$

and the Cauchy–Green strain measure  $C = F^t \cdot F$  as

$$C = C_* + 2\varepsilon[\alpha_{1*}\xi(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + \alpha_{2*}\eta\mathbf{k}\mathbf{k}], \quad C_* = \alpha_{1*}^2(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + \alpha_{2*}^2\mathbf{k}\mathbf{k}.$$

Here we retain only linear terms in  $\varepsilon$ . From the last relation, we find the variation of the Cauchy–Green strain measure

$$\delta C = 2\alpha_{1*} \delta\xi (\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}), \quad (3.2)$$

since the quantities with the subscript “\*” and  $\eta$  are specified (their variations are equal to zero).

Because the external loads on the lateral surfaces of the bar are zero and because the stress field is uniform, only the axial components  $T = T^{33}\mathbf{k}\mathbf{k}$  of the true stress tensor are different from zero. In view of relations (3.1), from this it follows that the second Piola–Kirchhoff tensor has the same representation  $P_{\Pi} = P_{\Pi}^{33}\mathbf{k}\mathbf{k}$  in both the current and intermediate configurations. As a result, taking into account relation (3.2), the first variational equation in (2.1) reduces to the equation  $(P_{\Pi}^{11} + P_{\Pi}^{22})\delta\xi = 0$ , which, by virtue of the arbitrariness of  $\delta\xi$  and with allowance for relation (2.2) and relation  $e_E = e - \beta\theta g$ , is written as

$$\begin{aligned} a_{11}\xi/\alpha_{1*} + a_{12}\theta &= b_1\eta/\alpha_{2*}, & a_{11} &= Q^{11}(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + Q^{22}(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}), \\ a_{12} &= (P_{\Pi,\Theta}^{11})_* + (P_{\Pi,\Theta}^{22})_* - \beta_*(Q^{11}(g) + Q^{22}(g)), \\ b_1 &= -[Q^{11}(\mathbf{k}\mathbf{k}) + Q^{22}(\mathbf{k}\mathbf{k})]. \end{aligned} \quad (3.3)$$

Here  $Q^{kl}(M)$  are the components of the second-rank tensor  $Q$  written in a Cartesian basis and dependent on the second-rank tensor  $M$ :  $Q(M) = J_* F_*^{-1} \cdot K(M) \cdot F_*^{-t}$ . The second-rank tensor  $K(M) = \tilde{L}_6^{IV} \cdot M$  is easy to calculate using relation (1.8), in which the tensor  $e_E$  is replaced by the corresponding tensor  $M$ , which is equal to  $\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}$ ,  $\mathbf{k}\mathbf{k}$  or the unit tensor  $g$ . In the case of an isothermal process, the second equation in (2.1) vanishes since the temperature (uniform) is specified (hence,  $\delta\Theta = 0$ ), and, in Eq. (3.3), only one unknown  $\xi$  is retained. If the process adiabatic and the temperature field is uniform, then,  $q_n = 0$  and  $\nabla\Theta = 0$  in the second equation in (2.1), and this equation becomes  $(A\dot{\Theta} + B\Theta - \rho_0\dot{\Omega})\delta\Theta = 0$ . Therefore, by virtue of the arbitrariness of  $\delta\Theta$ , the expression in brackets should vanish. Multiplying this expression by the small quantity  $\Delta t$  — the time of transition from the intermediate to the current configuration — and using expressions (1.13) for the coefficients  $A$  and  $B$ , the relationship between the small total, elastic, and temperature strains, and the relations  $\dot{\Theta}\Delta t = \varepsilon\theta$ ,  $D\Delta t = \varepsilon e$ , and  $\dot{T}\Delta t = \varepsilon\Gamma$  [ $\varepsilon\Gamma$  is the right side of expression (1.4), which does not contain  $T_*$ ], and  $\dot{\Omega}\Delta t = \varepsilon\omega$ , we obtain

$$\begin{aligned} a_{21}\xi/\alpha_{1*} + a_{22}\theta &= b_2\eta/\alpha_{2*} + \rho_0\omega, & a_{21} &= \Theta_*[\beta_*I_1(K(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j})) - (T_{,\Theta})_* \cdot (\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j})], \\ a_{22} &= \Theta_*[(\beta_{,\Theta})_*I_1(T_*) + 2\beta_*I_1((T_{,\Theta})_*) - \beta_*^2I_1(K(g)) - J_*^{-1}(W_{1,\Theta\Theta})_*] + J_*^{-1}\rho_0(c_T)_*, \\ b_2 &= -\Theta_*\{\beta_*[2T_*^{33} + I_1(K(\mathbf{k}\mathbf{k}))] - (T_{,\Theta})_* \cdot \mathbf{k}\mathbf{k}\}. \end{aligned} \quad (3.4)$$

Solving Eq. (3.3) in the  $(n+1)$ th step (in the case of an isothermal process) or system (3.3), (3.4) (in the case an adiabatic process) and knowing, as a result, the increments of the elongation  $\xi$  and temperature  $\theta$ , we determine the elastic and total site gradients from relations (1.3) and (3.1) the stress tensor and its derivative with respect to temperature from relation (1.4) and (1.7), and the functional  $W_1$  and its derivative with respect to temperature from relations (1.11) and (1.12); the obtained quantities will be the initial data for the next step.

In the numerical calculation, the initial temperature  $\Theta_0$  was set equal to 293 K. The material constants at this temperature were taken from [10–12] for 2959 grade rubber and had the following values:  $k_1 \stackrel{(1.9)}{=} k_{10} = 0.25$  MPa,  $k_2 \stackrel{(1.9)}{=} k_{20} = 0.25$  MPa,  $p_{10} = 1$ ,  $p_{20} = 0.425$ ,  $q_{10} = 374$ ,  $q_{20} = 300$ , and  $\chi_{20} \stackrel{(1.9)}{=} \chi_{200} = 770$  MPa. Here the notation  $\stackrel{(1.9)}{=}$  means “equal according to relation (1.9).” In the interval  $[\Theta_0, \Theta_0 + 100$  K], a linear temperature dependence of  $k_1$ ,  $k_2$ , and  $\chi_{20}$  is adopted. In the notation adopted in (1.9), this means that  $k_{11}(\Theta)$ ,  $k_{21}(\Theta)$ , and

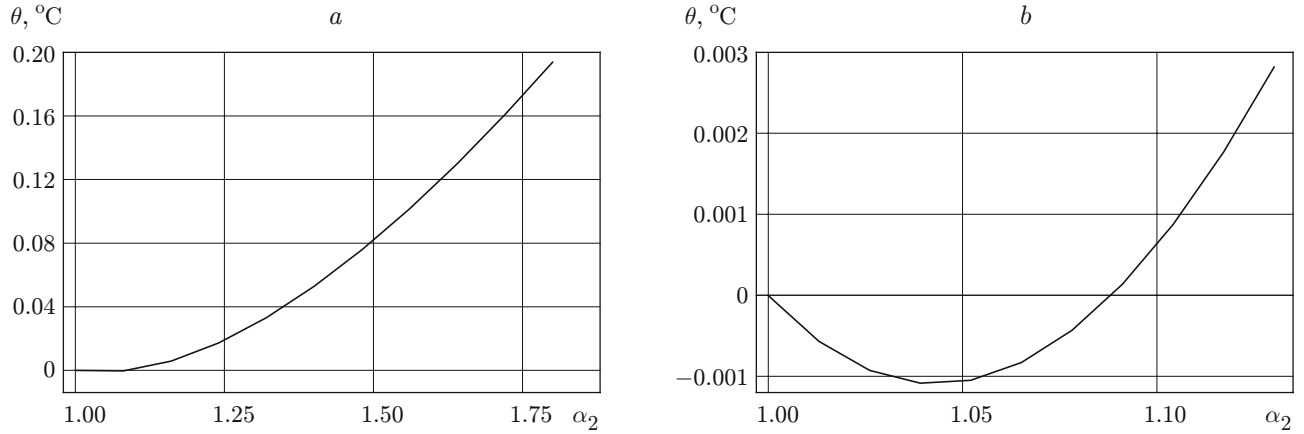


Fig. 1. Temperature variation for adiabatic extension of the bar: (a) general view; (b) initial portion of the curve shown in Fig. 1a.

$\chi_{201}(\Theta)$  are constants. The following values of these constants were assumed:  $k_{11} = k_{21} = 0.8 \cdot 10^{-3}$  MPa/K,  $\chi_{201} = -2$  MPa/K. Because the initial shear modulus  $G = 2(k_1 + k_2)$  is defined in terms of the constants  $k_1$  and  $k_2$ , it is linearly dependent on temperature, which agrees with both experimental data (see, for example, [13]) and the conclusions drawn from the propositions of statistical physics (see, for example, [14]):  $G = Nk\Theta$  ( $N$  is the number of chains in an elastomeric grid per unit volume and  $k$  is Boltzmann constant). The last relation is represented as  $G = G_0 + (G_0/\Theta_0)(\Theta - \Theta_0)$ , where  $G_0 = N\Theta_0$ . This relation is put in correspondence with the expression  $G = 2(k_{10} + k_{20}) + 2(k_{11} + k_{21})(\Theta - \Theta_0)$ . Taking into account the equality of the coefficients at  $\Theta - \Theta_0$  to the zero and first powers in these relation and setting  $k_{11} = k_{21}$ , we obtain the above values of  $k_{11}$  and  $k_{21}$  for the specified quantities  $k_{10}$  and  $k_{20}$ .

Experimental data on the temperature variation of the three-dimensional modulus for the same rubber are given in [15]. It has been shown that, as the temperature varies from  $\Theta_0$  to  $\Theta_0 + 110$  K, the three-dimensional modulus decreases by 28% and this variation can be considered linear within the confidence interval of the experiment. These data were used to determine the value of the coefficient  $\chi_{201}$ . In the temperature range considered, the linear temperature-expansion coefficient  $\beta$  was considered constant (data on its temperature dependence are not available) and equal to  $13.5 \cdot 10^{-5}$  K $^{-1}$  [16]. The initial density of the material was set equal to  $\rho_0 = 1.21 \cdot 10^3$  kg/m $^3$  [16]. The value of the specific (per unit mass) heat capacity at zero stress  $c_T$  is taken from a reference book [17]. The heat capacity was assumed to depend linearly on temperature in the range considered. In the notation used in relation (1.9), it was assumed that  $c_{T0} = 1.9 \cdot 10^{-3}$  MJ/(kg · K) and  $c_{T1} = 0.01 \cdot 10^{-3}$  MJ/(kg · K $^2$ ).

The bar was extended by a factor of 1.8 at a constant elongation rate of 0.1 sec $^{-1}$  within 1000 steps. Figure 1 shows the temperature variation during the adiabatic extension of the bar. The results given in Fig. 1 are in good agreement with the data of [18] both qualitatively and quantitatively. The quantitative agreement is apparently accidental since, in the present study and in [18], different materials were considered. A temperature decrease is usually associated with expansion of the sample, and its increase with a volume decrease (see, for example, [18]).

Figure 2 shows the relative volume variation  $\Delta V$  during adiabatic expansion of the bar. One can see that the temperature and volume variations are in good correlation.

The variation of the single component of the true stress tensor  $T^{33}$  during adiabatic extension of the bar is given in Fig. 3 (curve 1). Because of the small temperature variation, curve 1 almost coincides with the curve constructed by analytical solution of the problem of isothermal extension of the bar (see [9]).

The bar extension problem is of interest since it allows one to link the single nonzero stress to the free and internal energies, entropy, and temperature. Indeed, in the case of a thermoelastic process, the natural dissipation  $\varphi$  (see [4]) is represented as

$$T \cdot D = \rho(\dot{\Psi} + \dot{\Theta}s) \quad \text{or} \quad T \cdot D = \rho(\dot{u} - \Theta\dot{s}). \quad (3.5)$$

Here  $\Psi$ ,  $u$ , and  $s$  are the specific (per unit mass) free, internal energy, and entropy linked by the relation  $\Psi = u - \Theta s$ . The expression for the free energy obtained in [4] is written as  $\Psi = W_1/\rho_0 + \Psi_2(\Theta)$ , where the functional  $W_1$  is

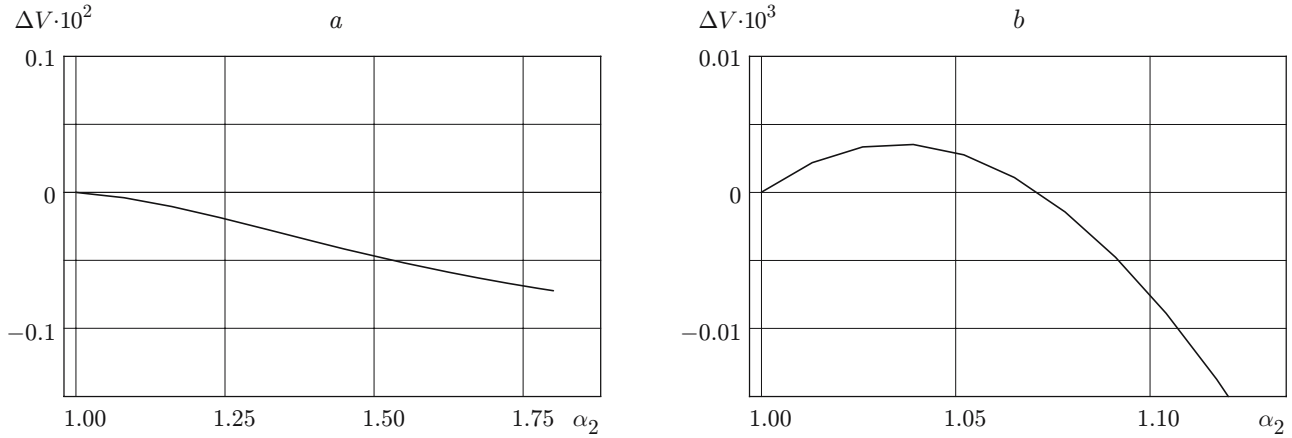


Fig. 2. Relative volume variation during adiabatic extension of the bar: (a) general view; (b) initial portion of the curve shown in Fig. 2a.

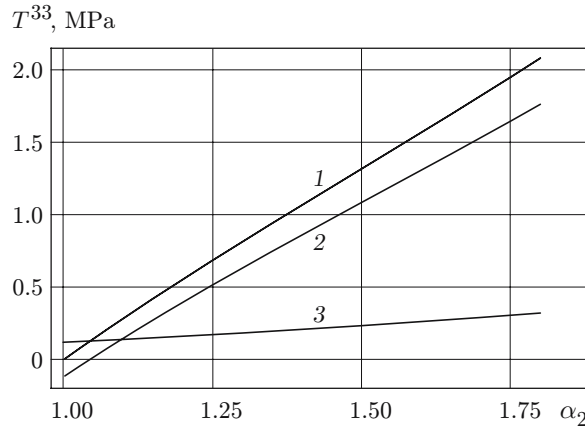


Fig. 3. Variation of the true-stress tensor component during adiabatic extension of the bar: curves 1–3 refer to  $T^{33}$ ,  $T_s$ , and  $T_u$ , respectively.

defined by relation (2.7) in [4] and the function  $\Psi_2(\Theta)$  by relation (2.9). Taking into account that  $\dot{W}_1 = JT \cdot D + \dot{\Theta}W_{1,\Theta} - \beta\dot{\Theta}JI_1(T)$  [the relation was obtained from (1.11) in the similar way as relation (1.14) was obtained from (1.12)] and using the expression for the entropy (relation (2.10) in [4]), we have

$$\dot{\Psi} = \rho_0 J^{-1} T \cdot D - s\dot{\Theta}, \quad \dot{u} = \rho_0 J^{-1} T \cdot D + \Theta\dot{s}. \quad (3.6)$$

Therefore, expressions (3.5) become identical equalities. Moreover, since in a thermoelastic adiabatic process, the entropy does not vary and it set equal to zero, from expressions (3.5) it follows that  $T \cdot D = \rho\dot{\Psi}$  in both isothermal and adiabatic processes, or  $T \cdot D = \rho\dot{u}$  in an adiabatic process. Therefore, the above relations become identities [in view of expressions (3.6) for such processes].

In the case of an isothermal process, the second relation in (3.5) provides an estimate for the contribution of the internal energy and entropy to the production of axial stress. According to (3.1), in the bar extension problem, the displacement rate gradient  $l = \dot{F} \cdot F^{-1}$ , whose symmetric part defines the strain rate tensor  $D$ , is represented as  $l = (\dot{\xi}/\alpha_1)(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + (\dot{\eta}/\alpha_2)\mathbf{k}\mathbf{k}$ . Setting  $\dot{\xi} \Delta t = \varepsilon\xi$  and  $\dot{\eta} \Delta t = \varepsilon\eta$ , from Eq. (3.3) in the case of an isothermal process, we obtain  $\dot{\xi} = \Lambda\dot{\eta}$ , where  $\Lambda = (\alpha_{1*}/\alpha_{2*})(b_1/a_{11})$ . Because  $l = D = \dot{h} = \dot{e}$  in the problem considered, we have

$$l = D = \dot{h} = \dot{e} = Z\dot{\eta}, \quad Z = (\Lambda/\alpha_1)(\mathbf{i}\mathbf{i} + \mathbf{j}\mathbf{j}) + (1/\alpha_2)\mathbf{k}\mathbf{k}. \quad (3.7)$$



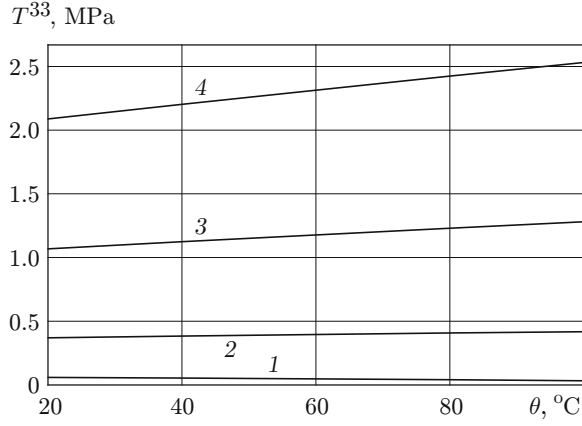


Fig. 4. Variation of the true axial stress during heating of a previously extended bar for  $\alpha_2 = 1.02$  (1), 1.13 (2), 1.4 (3), and 1.8 (4).

Then, the second relation in (3.5) is written as

$$T^{33}\dot{\eta} = \rho\alpha_2(\dot{u} - \Theta\dot{s}). \quad (3.8)$$

Using the equality  $\rho\dot{s} = B$  [ $B$  is the coefficient in (1.13)], which follows from relation (2.10) in [4] and is easily verified for an elastic material in the case of an isothermal process and taking into account (3.7), we obtain

$$\rho\Theta\dot{s} = \Theta B_0\dot{\eta}, \quad B_0 = \beta[2T \cdot Z + I_1(\tilde{L}_6^{IV} \cdot Z)] - T_{,\Theta} \cdot Z.$$

As a result, expression (3.8) has the form  $T^{33} = T_u + T_s$ , where  $T_u = \alpha_2(T \cdot Z + \Theta B_0)$  and  $T_s = -\alpha_2\Theta B_0$  are the contributions of the internal energy and entropy to the axial stress production. In Fig. 3, curve 3 corresponds to the quantity  $T_u$  and curve 2 to the quantity  $T_s$ . It is obvious that, in the case of an isothermal process, entropy makes the major contribution to the axial stress production. This is the well-known fact called entropic elasticity (see, for example, [18, 19]). In this case, an insignificant change in the internal energy allows it to be considered a function of only temperature (see, for example, [19]).

Further testing of the model was performed on the problem of heating of a previously extended bar. The bar was extended isothermally to a certain value  $\alpha_2$  in 1000 equal steps and then, also in 1000 equal steps, it was heated by  $100^\circ\text{C}$  at constant elongation (temperature increment in each step  $\theta = 0.08^\circ\text{C}$ ). In each step, we solved only Eq. (3.3) for the specified value of  $\theta$ . Figure 4 shows the variation of the true axial stress during the heating of the previously extended bar. With a rise in temperature, the stress in the bar decreases at low degrees of elongation and increases at high degrees. This effect is known as a temperature inversion (see [13, 14, 18]) and is attributed to two factors: an increase in the positive temperature strain upon heating, resulting in a decrease in the stress in the extended bar, and an increase in the shear modulus with a temperature rise, resulting in an increase in the tension stress. The interplay of these factors is easy to show by a simple engineering calculation assuming small strains.

The single nonzero axial stress in the bar is  $T = E(\Theta)(e - e_\Theta)$ . Here  $E$  is Young's modulus,  $e$  is the axial component of the total strain tensor,  $e_\Theta = \beta\Delta\Theta$  is the axial component of the temperature strain tensor, and  $\Delta\Theta = \Theta - \Theta_0$ . Setting  $E(\Theta) = E_0 + E_1\Delta\Theta$  and taking into account that  $E = 3G$  for an incompressible material, we obtain  $T = T_0 + 3[G_1e - \beta(G_0 + G_1\Delta\Theta)]\Delta\Theta$ , where  $T_0 = 3G_0e$ . An increase or decrease in the stress for positive  $\Delta\Theta$  depends on the sign of the expression in square brackets. After the substitution of the above values of material constants into the last relation, the sign of the expression in square brackets is determined by the quantity  $e - 4.2 \cdot 10^{-2}(1 + 3.2 \cdot 10^{-3}\Delta\Theta)$ . Consequently, if  $e < 0.042$ , which corresponds to  $\alpha_2 = \sqrt{1 + 2e} < 1.041$ , the stress decreases for  $\Delta\Theta > 0$ , and, if  $e > 0.0554$ , which corresponds to  $\alpha_2 > 1.054$ , the stress increases for  $0^\circ\text{C} < \Delta\Theta < 100^\circ\text{C}$ . For  $0.0420 < e < 0.0554$ , the stress first increases and then decreases in the indicated temperature range. The calculation results for small strains (curves 1 and 2) given in Fig. 4 are in good agreement with the estimates obtained. For curves 1 and 2, the stresses  $T_0$  calculated by the longitudinal-elasticity relation given above are equal to 0.06 and 0.42 MPa, respectively, which also agrees with the data in Fig. 4.

**Conclusions.** Thus, the thermoelastic equation of state was obtained using the formalized approach to constructing constitutive relations for complex media at finite strains. The kinematics of the process was determined by the superposition of small temperature strains and small elastic strains on finite thermoelastic strains. A general representation of the constitutive relation in increments and its exact evolutionary analog were obtained. Particular equations of state obtained using the elastic law for a slightly compressible material were given.

This work was supported by the Russian Foundation for Basic Research (Grant No. 06-01-00050), Program of the President of the Russian Federation on the State Support of Leading Scientific Schools (Grant No. NSH-8055.2006.1), a Program of the Russian Academy of Sciences Division of Energetics, Machine Engineering, Mechanics, and Control Processes, and an Integration Program of the Ural Division, Siberian Division, and Far East Division of the Russian Academy of Sciences.

## REFERENCES

1. R. S. Novokshanov and A. A. Rogovoi, "On constructing evolutionary constitutive relations for finite strains," *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, No. 4, 77–95 (2002).
2. R. S. Novokshanov and A. A. Rogovoi, "Evolutionary constitutive relations for finite viscoelastic strains," *Izv. Ross. Akad. Nauk., Mekh. Tverd. Tela*, No. 4, 122–140 (2005).
3. A. A. Rogovoi, "Constitutive relations for finite elastic–inelastic strains," *J. Appl. Mech. Tech. Phys.*, **46**, No. 5, 730–739 (2005).
4. A. A. Rogovoi, "Thermodynamics of finite-strain elastic–inelastic deformation," *J. Appl. Mech. Tech. Phys.*, **48**, No. 4, 591–598 (2007).
5. A. A. Rogovoi, "Kinematics of finite strain elastic–inelastic deformation," *J. Appl. Mech. Tech. Phys.*, **49**, No. 1, 136–141 (2007).
6. L. Vujošević and V. A. Lubarda, "Finite-strain thermoelasticity based on multiplicative decomposition of deformation gradient," *Theoret. Appl. Mech.*, **28/29**, 379–399 (2002).
7. V. A. Lubarda, "Constitutive theories based on the multiplicative decomposition of deformation gradient: Thermoelasticity, elastoplasticity, and biomechanics," *Appl. Mech. Rev.*, **57**, No. 2, 95–108 (2004).
8. A. Meggyes, "Multiple decomposition in finite deformation theory," *Acta Mech.*, **146**, 169–182 (2001).
9. V. V. Moshev, A. L. Svistkov, O. K. Garishin, et al., *Structural Mechanisms Involved in the Development of Mechanical Properties of Granular Polymer Composites* [in Russian], Inst. of Continuum Mechanics, Ekaterinburg (1997).
10. V. G. Kuznetsova and A. A. Rogovoi, "Effect of accounting for the slight compressibility of material in finite-strain elastic problems," *Izv. Ross. Akad. Nauk*, No. 4, 64–77 (1999).
11. V. G. Kuznetsova and A. A. Rogovoi, "Effect of accounting for the slight compressibility of elastomers. Axisymmetric problem. Analytical solution," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 6, 27–37 (2000).
12. A. Rogovoy, "Effect of elastomer slight compressibility," *Eur. J. Mech., A: Solids*, **20**, 757–775 (2001).
13. M. A. Mohsin, J. P. Berry, L. R. G. Treloar, "An experimental study of the thermodynamics of rubber in extension and torsion," *Brit. Polymer J.*, **18**, No. 3, 145–150 (1986).
14. R. E. Lyon and R. J. Farris, "Thermomechanics of rubber at small strains," *Polymer*, **28**, June, 1127–1132 (1987).
15. V. N. Zotin and V. N. Kovrov, "Effect of temperature on the compressibility of IRP-1226 rubber," in: *Rheological Behavior of Deformable Continua* (collected papers) [in Russian], Izd. Ural. Otd. Akad. Nauk SSSR, Sverdlovsk (1990), pp. 77–78.
16. A. P. Babichev, N. A. Babushkina, A. M. Bratkovskii, et al., *Physical Quantities* [in Russian], Énergoatomizdat, Moscow (1991).
17. A. I. Ivanchenko, V. A. Pakharenko, V. P. Privalko, et al., *Thermal and Rheological Characteristics of Polymers* [in Russian], Naukova Dumka, Kiev (1977).
18. L. R. G. Treloar, *The Physics of Rubber Elasticity*, Oxford University Press (2005).
19. V. G. Karnaukhov, *Coupled Problems of Thermoviscoelasticity* [in Russian], Naukova Dumka, Kiev (1982).